

Instanton Counting and Matrix Model

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Abstract

We construct an Imbimbo-Mukhi type matrix model, which reproduces exactly the partition function of \mathbb{CP}^1 topological strings in the small phase space, Nekrasov's instanton counting in $\mathcal{N} = 2$ gauge theory and the large N limit of the partition function in 2-dimensional Yang-Mills theory on a sphere. In addition, we propose a dual Stieltjes-Wigert type matrix model, which emerges when all-genus topological string amplitudes on certain simple toric Calabi-Yau manifolds are compared with the Imbimbo-Mukhi type model.

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1 Introduction

Recently, classical geometries prove to be very powerful in determining strong coupling gauge theory dynamics. A familiar story is the AdS/CFT correspondence in $\mathcal{N} = 4$ SYM[1]. Similarly, computing the confining phase superpotential in $\mathcal{N} = 1$ gauge theory has been facilitated by a large N duality for conifolds, called the geometric transition¹. On the dual deformed conifold, period integrals over holomorphic cycles are responsible for the effective superpotential of gluino condensates[3, 4, 5]. In this $\mathcal{N} = 1$ context, Dijkgraaf and Vafa conjectured that the all-genus free energy of certain target CY's, locally diffeomorphic to T^*S^3 , reduces to a hermitian (DV) matrix model[6, 7, 8]. As a further development of this celebrated matrix/geometry correspondence, for more general non-compact CY's, Aganagic et al. provided an unifying way in [9], where the \mathcal{W} -algebra symmetry (holomorphic diffeomorphism of CY's) in disguise is fully exploited to solve the all-genus free energy. In terms of the matrix model, these are known as \mathcal{W} -constraints on the τ -function of integrable hierarchies. Adopting the viewpoint that the partition function on CY 3-folds can be regarded as a multi-point function of non-compact B-branes, we can find its connection to the matrix model τ -function via the Frobenius-Kontsevich-Miwa transformation.

Our situation here is that the full non-perturbative partition function of $\mathcal{N} = 2$ Seiberg-Witten theory² is found to be captured by an Imbimbo-Mukhi³ type matrix model in the 't Hooft limit. Alternatively, because this instanton partition function[11, 12, 13] is precisely that of \mathbb{CP}^1 topological strings in the small phase space (see, e.g. [14, 15]), our proposal may serve as another sort of matrix/geometry correspondence, though now the integrable structure is quite simple.

The outline of this article is as follows. In the next section, we review the derivation of the Gromov-Witten prepotential of \mathbb{CP}^1 topological strings in the small phase space. We focus on its free fermion representation. Though the integrable structure is simple, it is still fruitful enough to exhibit the underlying Toda hierarchy. In sec.3, we construct an Imbimbo-Mukhi type matrix model to compute the \mathbb{CP}^1 partition function and sketch a proof using essentially properties of Schur polynomials. Meanwhile, our result gives spontaneously Nekrasov's instanton counting in $\mathcal{N} = 2$ gauge theory. Comments on relations to 2-dimensional Yang-Mills theory are summarized in sec.4. Based on observations of all-genus amplitudes of topological strings on some simple toric CY's, we also propose a Stieltjes-Wigert type matrix model dual to the previous Imbimbo-Mukhi type one in the 't Hooft limit.

¹See [2] for a recent review.

²It can be either abelian or non-abelian. The former case is realized by wrapping a D5-brane on a $\mathbb{CP}^1 \subset \mathbf{K3}$, where the non-perturbative effect arises from worldsheet instantons. By rearranging the abelian partition function, one as well obtains a non-abelian version, see Appendix B.

³The Imbimbo-Mukhi matrix model[10] is a τ -function satisfying the 2-time Toda lattice hierarchy.

2 τ -function of \mathbb{CP}^1 model

The complex projective space \mathbb{CP}^1 is the simplest target manifold with a positive first Chern class. Since \mathbb{CP}^1 has only two de Rham classes, i.e. the identity and the Kähler class, the topological A-model on it will have two corresponding physical observables, i.e. P and Q , respectively. Note that $t_{n,P}$ and $t_{n,Q}$ ($n > 0$) indicating couplings to gravitational descendants of P and Q are identified with times of the Toda hierarchy in [15, 16]. When all couplings are turned off (in the small phase space), the (genus zero) free energy is

$$\frac{1}{2}t_{0,P}^2 t_{0,Q} + e^{t_{0,Q}}, \quad (2.1)$$

where $t_{0,Q} := t$ denotes the Kähler parameter of \mathbb{CP}^1 and $t_{0,P}$ will play the role of the complex scalar vev in the $\mathcal{N}=2$ $U(1)$ vector multiplet. Besides, the exponential term comes from the degree-1 instanton.

Following [14, 17], the partition function (τ -function) can be expressed as a sum over Young tableaux $\lambda = (k_1, \dots, k_j, \dots)$ (k_i : i -th row length, $|\lambda|$: number of boxes):

$$Z^{CP^1} = \sum_{\lambda} \frac{\mathbf{m}_{\lambda}^2}{\hbar^{2|\lambda|}} \exp\left(t \frac{\text{ch}_2(a, \lambda)}{2\hbar^2}\right), \quad a = t_{0,P}, \quad (2.2)$$

$$\mathbf{m}_{\lambda} = \prod_{\square} \frac{1}{h(\square)} = \frac{d_{\lambda}}{|\lambda|!} = \prod_{i < j} \frac{k_i - k_j + j - i}{j - i}, \quad h(\square) : \text{hook length}.$$

Here, \hbar denotes the genus expansion parameter. Note that \mathbf{m}_{λ} and d_{λ} , respectively, are the Plancherel measure and the dimension of λ as a representation of S_{λ} (symmetric group). In addition, the Chern polynomial Ch_n is defined as a Fourier transform of the second derivative of $f_{\lambda}(x, a)$, which is the *profile function* obtained by rotating counterclockwise the Young tableau by $\frac{4}{3}\pi$:

$$\frac{1}{2} \int dx f_{\lambda}''(x, a) e^{ux} = \sum_{n=0}^{\infty} \frac{u^n}{n!} \text{ch}_n(a, \lambda), \quad (2.3)$$

where (box size: $\hbar \times \hbar$)

$$\begin{aligned} f_{\lambda}(x) &= |x - a| + \rho(x), \\ \rho(x) &= \sum_{i=1}^{\infty} \left(|x - a - \hbar k_i + \hbar(i-1)| - |x - a - \hbar k_i + \hbar i| \right. \\ &\quad \left. + |x - a + \hbar i| - |x - a + \hbar(i-1)| \right). \end{aligned} \quad (2.4)$$

Alternatively, Z^{CP^1} can be written into a more compact matrix element form:

$$Z^{CP^1} = \langle 0 | e^{\frac{J}{\hbar}} e^{tL_0} e^{\frac{J-1}{\hbar}} | 0 \rangle. \quad (2.5)$$

To explain the notation, let us make use of fermionic operators b, c satisfying the usual properties:

$$\{b_r, c_s\} = \delta_{r+s,0}, \quad r, s \in \mathbb{Z} + 1/2; \quad b_r|0\rangle = c_s|0\rangle = 0, \quad \forall r, s > 0 \quad (2.6)$$

to define an orthogonal basis $\{|\lambda\rangle\}$ w.r.t. λ :

$$|\lambda\rangle = \prod_{i=1}^{\infty} b_{i-k_i-\frac{1}{2}}|0'\rangle, \quad \langle\lambda|\kappa\rangle = \delta_{\lambda,\kappa}; \quad c_r|0'\rangle = 0, \quad \forall r, \quad (2.7)$$

which satisfies

$$e^{\frac{J_{-1}}{\hbar}}|0\rangle = \sum_{\lambda} \frac{1}{\hbar^{|\lambda|} \prod_{\square \in \lambda} h(\square)} |\lambda\rangle \quad (2.8)$$

and

$$L_0|\lambda\rangle = \frac{1}{2\hbar^2} \text{ch}_2(a, \lambda) |\lambda\rangle. \quad (2.9)$$

Note that the Virasoro generator $L_0 = \frac{1}{2}J_0^2 + J_{-1}J_1$ consists of the $U(1)$ current defined using the bosonization:

$$J(z) =: b(z)c(z) := \sum_{n \in \mathbb{Z}} J_n z^{-n-1}, \quad J_n = \sum_{s \in \mathbb{Z}+1/2} : b_s c_{n-s} :, \quad [J_m, J_{-n}] = n\delta_{m+n,0}. \quad (2.10)$$

For $J_0 = \frac{a}{\hbar}$, the matrix element reads

$$Z^{CP^1} = \exp \frac{1}{\hbar^2} \left(e^t + \frac{1}{2} t a^2 \right). \quad (2.11)$$

It can be verified that ($\hbar = 1$)

$$\frac{\partial^2}{\partial t^2} \log Z^{CP^1}(a) = \frac{Z^{CP^1}(a-1) Z^{CP^1}(a+1)}{\left(Z^{CP^1}(a) \right)^2}, \quad (2.12)$$

which is none other than the first equation of the Toda lattice hierarchy.

3 Imbimbo-Mukhi type matrix model

Before writing down our matrix model, let us first study an illuminating generalization, namely, the Imbimbo-Mukhi matrix model which describes the deformed conifold T^*S^3 with B-brane insertions.

According to [9], on the B-model side, to vary the complex structure of CY's is done by wrapping non-compact B-branes on degenerating fibers. The problem hence boils down to dealing with the moduli space of a Riemann surface. Moreover, with the special geometry relation, quantizing the complex moduli is realized by introducing quantum

Kodaira-Spencer fields. The vacuum expectation value of the KS field in turn defines a *quantum* Riemann surface which determines the CY with varied complex moduli. For full details on this subject, readers are recommended to consult [9].

We quickly review the case of T^*S^3 , whose defining equation is

$$uv - H(x, y) = 0, \quad H(x, y) = xy - \mu. \quad (3.1)$$

Two KS fields (chiral bosons)

$$x = \partial_y \tilde{\phi}(y), \quad y = -\partial_x \phi(x) \quad (3.2)$$

are defined on asymptotic regions $x, y \rightarrow \infty$ of the degenerating locus $H(x, y) = 0$ which has a topology of the sphere. If there are initially N non-compact B-branes at x -patch

$$\langle N | \psi(x_1) \cdots \psi(x_N) | W \rangle, \quad \psi = e^{\frac{1}{g_s} \int y}, \quad (3.3)$$

where $|W\rangle \in \mathcal{H}$ (\mathcal{H} : Hilbert space of chiral bosons ϕ) and $\langle N |$ denotes an N -fermion state, the complex structure of T^*S^3 is deformed to be

$$xy = \mu \longrightarrow x(y + \sum_{n>0} n t_n x^{n-1}) = \mu, \quad t_n = -\frac{g_s}{n} \sum_{i=1}^N x_i^{-n}. \quad (3.4)$$

Notice that the canonical transformation relating x -patch and y -patch gives rise to the S -matrix, which acts on the B-brane operator as a Fourier transform, i.e.

$$\tilde{\psi}(y) = S\psi(y) = \frac{1}{\sqrt{2\pi g_s}} \int dx e^{-\frac{xy}{g_s}} \psi(x). \quad (3.5)$$

In the deformed geometry (3.4), moving B-branes from x -patch to y -patch $Y = \text{diag}(y_1, \dots, y_N)$ leads to

$$\int \prod_{i=1}^N dx_i e^{-\sum_{i=1}^N x_i y_i / g_s} \langle \psi(x_1) \cdots \psi(x_N) \rangle. \quad (3.6)$$

Further, via the Harish-Chandra-Itzykson-Zuber formula

$$\int DU e^{i \text{Tr}(BX)} \propto \frac{\det e^{ix_j p_j}}{\Delta(x) \Delta(p)}, \quad \Delta : \text{VanderMonde}, \quad (3.7)$$

where $X = \text{diag}(x_1, \dots, x_N)$ and $B = U^\dagger P U$ with $P = \text{diag}(p_1, \dots, p_N)$, we are able to obtain a Kontsevich-like matrix model for (3.6), i.e.

$$\begin{aligned} Z(\tilde{t}, t) &= \frac{1}{\Delta(y)} \int \prod dx_i \Delta(x) (\det X)^{-\mu/g_s} e^{\frac{1}{g_s} (\sum_n t_n \text{Tr} X^n + \sum_i x_i y_i)} \\ &= \int_{N \times N} dX (\det X)^{-\mu/g_s} e^{\frac{1}{g_s} \text{Tr} (\sum_n t_n \text{Tr} X^n + XY)}, \end{aligned} \quad (3.8)$$

where $\tilde{t}_n = \frac{g_s}{n} \text{Tr} Y^{-n}$. For large enough N , we choose $\mu = g_s N$. (3.8) is the so-called Imbimbo-Mukhi matrix model first derived in [10].

3.1 Proof

We are now in a position to propose our model. Under the 't Hooft limit:

$$\mu = Ng_s : \text{large but fixed}, \quad N \rightarrow \infty, \quad g_s \rightarrow 0, \quad (3.9)$$

the following hermitian matrix integral reproduces Nekrasov's instanton counting formula as well as the previous Z^{CP^1} , i.e.

$$\mathbf{Z}(\mathbf{U}) = \int_{N \times N} dX (\det X)^{-\mu} \exp \text{Tr} \mu \left(\sum_{n>0} \mathbf{K}_n X^n + X \mathbf{U}^{-1} \right), \quad (3.10)$$

where

$$\mu \mathbf{K}_n = -\frac{1}{n} \text{Tr} \mathbf{U}^n, \quad \mathbf{U} = \text{diag} (q^{\frac{1}{2}}, \dots, q^{N-\frac{1}{2}}), \quad q = e^{-g_s}. \quad (3.11)$$

Let us now sketch a proof of our proposal. A fact of particular use is that the τ -function (3.8) admits a S -matrix formulation[18], where the S -matrix has been explicitly diagonalized thanks to tachyon reflection amplitudes in the bosonic $c = 1$ string. Based on those, we rewrite $\mathbf{Z}(\mathbf{U})$ into (S : S -matrix)

$$\mathbf{Z}(\mathbf{U}) = \langle \ell | S | \ell \rangle, \quad | \ell \rangle = \exp \sum_{n=1}^{\infty} \frac{\text{Tr} \mathbf{U}^n}{n} \alpha_{-n} | 0 \rangle, \quad \ell_n = \text{Tr} \mathbf{U}^n. \quad (3.12)$$

Given all mathematical details in Appendix A, we insert the identity operator $1 = \sum_R |R\rangle \langle R|$ into (3.12) to get

$$\mathbf{Z}(\mathbf{U}) = \sum_R \mathcal{N}_R s_R(\ell) s_R(\ell), \quad \mathcal{N}_R = \prod_{\square(i,j)} (\mu - i + j), \quad (3.13)$$

where $s_R(\ell)$ is the Schur polynomial and the diagonal element \mathcal{N}_R can be found in [19]⁴. Upon imposing the 't Hooft limit, $\mathbf{Z}(\mathbf{U})$ reduces to a MacMahon function[26, 27]⁵:

$$\mathbf{Z}(\mathbf{U}) = \sum_R \mu^{|R|} s_R(\ell)^2 = \prod_{n=1}^{N=\infty} \frac{1}{(1 - \mu q^n)^n}. \quad (3.14)$$

We have taken advantage of the Cauchy's formula

$$\sum_{\lambda} \text{Tr}_{\lambda} \mathbf{A} \text{Tr}_{\lambda} \mathbf{B} = \prod_{i,j} \frac{1}{1 - a_i b_j}, \quad (3.15)$$

where a_i (b_i) denotes the eigenvalue of \mathbf{A} (\mathbf{B}). Another way defining the Schur polynomial in terms of q is

$$s_R(\ell) = \text{Tr}_R \mathbf{U} = q^{-\frac{1}{2} \sum_{(i,j) \in \lambda} (j-i)} \prod_{(i,j) \in \lambda} \frac{1}{q^{-h(i,j)/2} - q^{h(i,j)/2}}, \quad h(i,j) : \text{hook length}. \quad (3.16)$$

⁴Note that the diagonal element \mathcal{N}_R also appears in the context of 1/2 BPS correlators in $\mathcal{N} = 4$ SYM, see [20, 21].

⁵The subleading term $\frac{\kappa(R)}{2} \mu^{|R|-1}$ in \mathcal{N}_R , where $2 \sum_{\square(i,j)} (-i + j) = \kappa(R)$, is omitted when compared to the leading one.

By expanding q in (3.16) around small g_s , together with (2.2), it is seen that

$$\sum_R \mu^{|R|} s_R(\ell)^2 \rightarrow \sum_R \left(\frac{\mu}{g_s^2}\right)^{|R|} \mathbf{m}_R^2. \quad (3.17)$$

Amazingly, the partition function of \mathbb{CP}^1 in the small phase space is reproduced upon

$$g_s \rightarrow \hbar, \quad \mu \rightarrow e^t + \frac{1}{2}a^2 t. \quad (3.18)$$

It is fine that a can be turned off to zero because it stands for the complex scalar vev in the $\mathcal{N}=2$ $U(1)$ vector multiplet below.

3.2 Nekrasov's formula

As is shown, our Imbimbo-Mukhi type matrix model proves to give

$$\sum_{\lambda} \left(\frac{\mu}{\hbar^2}\right)^{|\lambda|} \mathbf{m}_{\lambda}^2 \quad (3.19)$$

under the 't Hooft limit. As a matter of fact, (3.19) is nothing but Nekrasov's partition function in $\mathcal{N} = 2$ $U(1)$ gauge theory[14]⁶:

$$Z^{inst}(a, \Lambda, \hbar) = \sum_{\lambda} \exp(-\mathcal{E}[f, \lambda]), \quad (3.20)$$

$$\mathcal{E}[f, \lambda] = \frac{1}{2} \int \int_{x_1 > x_2} dx_1 dx_2 f_{\lambda}''(x_1) f_{\lambda}''(x_2) \gamma_{\hbar}(x_1 - x_2, \Lambda),$$

where

$$\gamma_{\hbar}(x, \Lambda) := \frac{d}{ds} \Big|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^{\infty} \frac{dt}{t^{1-s}} \frac{e^{-tx}}{(e^{\hbar t} - 1)(e^{-\hbar t} - 1)}, \quad (3.21)$$

$$\Gamma(s) = \int_0^{\infty} dt e^{-t} t^{s-1}.$$

In addition, $\gamma_{\hbar}(x, \Lambda)$ has a *genus* expansion w.r.t. \hbar :

$$\begin{aligned} \gamma_{\hbar}(x, \Lambda) &= \sum_{g=0}^{\infty} \hbar^{2g-2} \gamma_g(x, \Lambda) \\ &= \frac{1}{\hbar^2} \left(\frac{1}{2} x^2 \log\left(\frac{x}{\Lambda}\right) - \frac{3}{4} x^2 \right) - \frac{1}{12} \log \frac{x}{\Lambda} + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)!} \left(\frac{\hbar}{x}\right)^{2g-2}, \end{aligned} \quad (3.22)$$

where an identity

$$\frac{1}{(e^t - 1)(e^{-t} - 1)} = \frac{d}{dt} \frac{1}{e^t - 1} = \frac{1}{t^2} + \sum_{g=1}^{\infty} \frac{B_{2g}}{2g(2g-2)!} t^{2g-2} \quad (3.23)$$

⁶How to convert (3.19) into a non-abelian instanton partition function is summarized in Appendix B.

and the definition of the Bernoulli number

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \quad (3.24)$$

have been used. Through the substitution

$$\mu = \frac{1}{2} a^2 \log \Lambda + \Lambda, \quad (3.25)$$

we yield[14]

$$Z^{inst}(a, \Lambda, \hbar) = \sum_{\lambda} \left(\frac{\mu}{\hbar^2} \right)^{|\lambda|} \mathbf{m}_{\lambda}^2. \quad (3.26)$$

Notice again that choosing $a = 0$ does not lose any generality.

4 Comments

So far, we have constructed an Imbimbo-Mukhi type matrix model. In the 't Hooft limit, it reproduces the partition function of \mathbb{CP}^1 topological strings in the small phase space. Besides, it is none other than Nekrasov's instanton counting formula in this limit.

Let us include two more applications. The first is that our model can recover the partition function of 2-dimensional Yang-Mills theory on a sphere. Another aspect is that the all-genus free energy of special toric Calabi-Yaus can be derived from our model. To this end, we further propose a dual Stieltjes-Wigert type matrix model.

4.1 2-dimensional Yang-Mills

The partition function of 2d $U(N)$ YM on a genus g and area A orientable manifold is[22, 23]

$$Z^{2d} = \sum_R (\dim R)^{2-2g} \exp \left(- \frac{\lambda A}{2N} C_2(R) \right), \quad (4.1)$$

where λ is the 't Hooft coupling of 2d YM, $C_2(R)$ is the quadratic Casimir of the representation R (conventions inherited from (2.2) and [38]; the box size is $\hbar \times \hbar$)

$$C_2(R) = \sum_{i=1}^N \left(N k_i + k_i(k_i - 2i + 1) \right) = \int dx f_R''(x) \left(\frac{N}{4\hbar^2} x^2 + \frac{1}{6\hbar^3} x^3 \right) \quad (4.2)$$

and

$$\begin{aligned} \dim R &= \prod_{1 \leq i < j \leq N} \frac{k_i - k_j + j - i}{j - i} \\ &= \exp \left(- \frac{1}{4} \int \int_{x > y} dx dy f_R''(x) f_R''(y) \gamma_{\hbar}(x - y) + \frac{1}{2} \int dx f_R''(x) \left(\gamma_{\hbar}(x + \hbar N) - \gamma_{\hbar}(\hbar N) \right) \right). \end{aligned} \quad (4.3)$$

Here, we have set $a = 0$ and $\Lambda = 1$ for $f_R(x, a)$ and $\gamma_h(x, \Lambda)$. Note that the second term in the above second line serves as a cut-off which will be dropped out when N is taken to infinity.

At the first sight, when N goes to infinity, $\dim R \rightarrow \mathbf{m}_R$ and $C_2(R) \sim N|R|$, so the genus zero Z^{2d} is exactly of the form (3.19). Let us check this point precisely. Combining everything, we yield

$$\begin{aligned} Z^{2d} = \sum_R \exp \left(-\frac{2-2g}{4} \int \int_{x>y} dx dy f_R''(x) f_R''(y) \gamma_h(x-y) \right. \\ \left. + \frac{2-2g}{2} \int dx f_R''(x) \left(\gamma_h(x+\hbar N) - \gamma_h(\hbar N) \right) - \frac{\lambda A}{4N} \int dx f_R''(x) \left(\frac{N}{4\hbar^2} x^2 + \frac{1}{6\hbar^3} x^3 \right) \right). \end{aligned} \quad (4.4)$$

For large N , we turn out to get

$$Z^{2d} \rightarrow \sum_R \exp \left(-\frac{1-g}{2} \int \int_{x>y} dx dy f_R''(x) f_R''(y) \gamma_h(x-y) - \frac{\lambda A}{8\hbar^2} \int dx f_R''(x) x^2 \right). \quad (4.5)$$

Utilizing another equality for the profile function $f_R(x)$ [12]:

$$|R| = \frac{1}{4\hbar^2} \int dx f_R''(x) x^2, \quad (4.6)$$

in the genus zero case, we are left with

$$Z^{2d} = \sum_R \exp \left(\frac{-\lambda A |R|}{2} \right) \frac{\mathbf{m}_R^2}{\hbar^2 |R|}. \quad (4.7)$$

In other words, we may have recovered the 2d YM partition function in large N limit using our matrix model upon identifying⁷ $\mu \equiv e^{-\lambda A/2}$.

4.2 Toric Calabi-Yau's and unknot Wilson loops

It is well-known that $U(N)$ level ℓ Chern-Simons gauge theory on S^3 is large N dual to the topological A-model on a resolved conifold $\mathcal{O}(-1) + \mathcal{O}(-1) \rightarrow \mathbb{P}^1$. That is, the partition function on the resolved conifold (Kähler parameter $t = g_s N$, $q = e^{-g_s}$)

$$Z_{P^1} = \exp(-\mathcal{F}), \quad \mathcal{F} = \sum_{n>0} \frac{e^{-nt}}{n \left(2 \sin(n g_s/2) \right)^2}, \quad g_s = \frac{2\pi i}{\ell + N} \quad (4.8)$$

is that of Chern-Simons under the 't Hooft expansion [24, 25]. This is an open/closed duality realized via the geometric transition of conifolds.

⁷Note that both μ and $e^{-\lambda A/2}$ can be complex. For μ , this is true in the context of the CY manifold. In the case of $-\lambda A/2$, it plays the role of the gauge coupling of a 4-dimensional theory (e.g. D5-branes compactified on the S^2 of a resolved conifold) and can be complexified by turning on the theta angle.

As another example[26], let us consider two $U(N)$ Chern-Simons on two S^3 's, separated from each other by a complexified Kähler parameter t' . The Ooguri-Vafa operator[28] associated with two holonomies in this setup is

$$Z(U, V, t') = \exp \left(\sum_{n>0} \frac{1}{n} e^{nt'} \text{Tr } U^n \text{Tr } V^n \right) = \sum_R e^{-t'|R|} \text{Tr}_R U \text{Tr}_R V \quad (4.9)$$

with the normalized vev

$$\langle Z(U, V, t') \rangle = \sum_R e^{-t'|R|} W_R(\mathcal{K}) W_R(\mathcal{K}) = \sum_R M^{|R|} s_R(x_i = q^{\frac{1}{2}-i})^2, \quad (4.10)$$

$$W_R(\mathcal{K}) = s_R(x_i = q^{(N-2i+1)/2}), \quad M = e^{-t'} q^N.$$

Note that $W_R(\mathcal{K})$ stands for an unknot Wilson loop in S^3 . After the geometric transition, a smooth toric CY emerges which is characterized by three Kähler parameters, namely, $t_1 = t_2 = g_s N$ and $t = t' - (\frac{t_1+t_2}{2})$. The free energy $-\log \langle Z(U, V, t') \rangle$ is

$$\mathcal{F} = \sum_{n>0} \frac{e^{-nt_1} + e^{-nt_2} + e^{-nt}(1 - e^{-nt_1})(1 - e^{-nt_2})}{n \left(2 \sin(n g_s / 2) \right)^2}. \quad (4.11)$$

It is found that (4.10) readily agrees with our result in (3.17) if $q \rightarrow q^{-1}$ and $M \rightarrow \mu$ are carried out in (4.10).

4.3 Stieltjes-Wigert type matrix model

Inspired by (4.11), we are capable of proposing a Stieltjes-Wigert type matrix model which is dual to (3.10). That is,

$$Z_{SW} = \frac{1}{\text{vol} U(N)} \int_{N \times N} dY \exp \text{Tr} \left(-\frac{1}{2g_s} (\log Y)^2 + \log(Y \otimes 1_{m \times m} - 1_{N \times N} \otimes \mathbf{V}) \right),$$

$$\mathbf{V} = \text{diag}(v_1, \dots, v_i, \dots), \quad i = 1, \dots, m, \quad (4.12)$$

where Y is hermitian and the measure gives the usual VanderMonde. Without the second logarithm term inside the potential, (4.12) is originally designated[29, 30, 31, 32]⁸ to rewrite the partition function of $U(N)$ Chern-Simons theory on S^3 .

To prove the equivalence to (3.10), we note that (4.12) dictates a B-model calculation of the non-compact B-brane correlator[35, 36] with their moduli \mathbf{V} . Put it differently, (4.12) is a mirror description of the A-model resolved conifold, on whose external leg of the toric diagram a stack of probe Lagrangian branes is inserted. Besides, each component

⁸ See [33, 34] for more details about Chern-Simons theory and Stieltjes-Wigert polynomials.

of \mathbf{V} is chosen to be $v_i = \exp -g_s(\ell - i + m + \frac{1}{2})^9$, which immediately implies that these Lagrangian branes are all located at the same place and, therefore, trigger a geometric transition such that a new toric CY appears, referred to as a bubbling CY in [37].

Translating everything into the A-model language, one can utilize the melting crystal method to write down the Lagrangian brane scattering amplitude via the quantum dilogarithm $L(v, q)$ as (Kähler parameter of \mathbb{P}^1 : $t = (N + m)g_s$)

$$Z_{crystal} = Z_{P^1}(t)M(q) \prod_{i < j} (1 - \frac{v_i}{v_j}) \prod_{i=1}^m \frac{L(v_i, q)}{L(v_i e^{-t}, q)}, \quad (4.13)$$

$$L(v, q) = \prod_{k=1}^{\infty} (1 - vq^k) = \exp \left(\sum_{k=1}^{\infty} \frac{v^k}{k[k]} \right), \quad [k] = q^{k/2} - q^{-k/2},$$

which is essentially the same as (4.12) up to an irrelevant overall factor and a framing choice[36]. Moreover, by doing so and inserting the moduli variable \mathbf{V} , it is seen that $-\log(Z_{crystal}/M(q)^2)$ coincides with (4.11), except now[37]

$$t_1 = g_s m, \quad t_2 = g_s(N - m), \quad t = g_s \ell. \quad (4.14)$$

In other words, given the result discussed in sec.4.2, the Stieltjes-Wigert type matrix model in (4.12) can be regarded to be equivalent to (3.10) (up to MacMahon's) in the 't Hooft limit through $m = \frac{N}{2} = N' \rightarrow \infty$, $q \rightarrow q^{-1}$ and $e^t \rightarrow \mu$.

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A Conventions

Since we heavily rely on the $U(\infty)$ representation theory and symmetric polynomials, let us explain briefly the notation following [26]. Recall that, via the Frobenius formula, the Schur polynomial labeled by a Young tableau R is expressed as (for an $N \times N$ unitary matrix U)

$$\langle R|U \rangle = \text{Tr}_R U = \sum_{\vec{k}} \frac{\chi_R(C(\vec{k}))}{z_{\vec{k}}} \Upsilon_{\vec{k}}(U) = \frac{\det u_j^{k_i + N - i}}{\det u_j^{N - i}}, \quad (A.1)$$

⁹As pointed out in [37], ℓ and m here assign an $U(N)$ representation via a rectangular Young tableau of m rows and ℓ columns.

where u_i 's are eigenvalues of U , while k_i is the i -th row length of R . In addition, $\chi_R(C(\vec{k}))$ is the character w.r.t. R evaluated at the conjugacy class $C(\vec{k})$ of the symmetric group S_l where

$$l = \sum_j k_j, \quad \vec{k} = (k_1, k_2, \dots) \quad (\text{A.2})$$

and

$$\Upsilon_{\vec{k}}(U) = \prod_{j=1}^{\infty} (\text{Tr } U^j)^{k_j}, \quad z_{\vec{k}} = \prod_j k_j! j^{k_j}. \quad (\text{A.3})$$

The character $\chi_R(C(\vec{k}))$ is referred to as $\langle R | \vec{k} \rangle$ if one quotes the 2d CFT techniques in (2.6) and (2.10) to define what follow:

$$\begin{aligned} |\vec{k}\rangle &= \prod_{j=1}^{\infty} (\alpha_{-j})^{k_j} |0\rangle, & |R\rangle &= \epsilon(R) \prod_{i=1}^r b_{-m_i - \frac{1}{2}} c_{-n_i - \frac{1}{2}} |0\rangle, \\ \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} |\vec{k}\rangle \langle \vec{k}| &= 1, & \langle R | R' \rangle &= \delta_{R, R'}, & \epsilon(R) &= \exp i\pi \left(\sum_{i=1}^r n_i + \frac{1}{2} r(r-1) \right), \end{aligned} \quad (\text{A.4})$$

where r is the number of diagonal boxes of R , while integers m_i and n_i ($i = 1, \dots, r$) are

$$m_i = h_i - i, \quad n_i = v_i - i. \quad (\text{A.5})$$

Here, h_i (v_i) labels the i -th row (column) length. These tools altogether facilitate another expression for a Schur polynomial, i.e.

$$s_R(\omega) = \text{Tr}_R \mathbf{W} = \langle R | \omega \rangle, \quad |\omega\rangle = \exp \sum_{n=1}^{\infty} \frac{\omega_n}{n} \alpha_{-n} |0\rangle, \quad \omega_n = \text{Tr } \mathbf{W}^n. \quad (\text{A.6})$$

B Non-abelian instanton counting

For $\mathcal{N} = 2$ $SU(m)$ gauge theory, the non-perturbative part of Nekrasov's formula is

$$Z^{\text{int}} = \sum_{\vec{R}} \Lambda^{|\vec{R}|} \prod_{(s,i) \neq (r,j)} \frac{a_s - a_r + \hbar(k_{s,i} - k_{r,j} + j - i)}{a_s - a_r + \hbar(j - i)}, \quad |\vec{R}| = \sum_{i=1}^m |\vec{R}_i|, \quad (\text{B.1})$$

where Λ is the dynamical scale and \vec{R} consists of m Young tableaux located at a_r 's[11]. According to [38], given an arbitrary Young tableau λ , we can divide it into m subparti-

tions $\subset \vec{R}$ and convert its Plancherel measure into (B.1) as follows:

$$\begin{aligned}
\frac{1}{\hbar^{2|\lambda|}} \mathbf{m}_\lambda^2 &= \lim_{N \rightarrow \infty} \prod_{i < j}^N \left(\frac{k_i - k_j + j - i}{j - i} \right)^2 \\
&= \prod_{s=1}^m \prod_{i \neq j}^{N_s} \frac{k_{s,i} - k_{s,j} + j - i}{j - i} \prod_{s \neq r}^m \prod_{i=1}^{N_s} \prod_{j=1}^{N_r} \frac{W_s - W_r + k_{s,i} - k_{r,j} + j - i}{W_s - W_r + j - i} \\
&= \prod_{(s,i) \neq (r,j)} \frac{a_s - a_r + \hbar(k_{s,i} - k_{r,j} + j - i)}{a_s - a_r + \hbar(j - i)}, \\
\hbar W_s &= \hbar(N - (N_1 + \dots + N_{s-1})) = a_s,
\end{aligned} \tag{B.2}$$

where a_r 's denote vevs of complex scalars in vector multiplets of unbroken $U(1)^{m-1} \subset SU(m)$.

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